Robustness and Recourse for the single period Open-Pit Problem with Processing Decision

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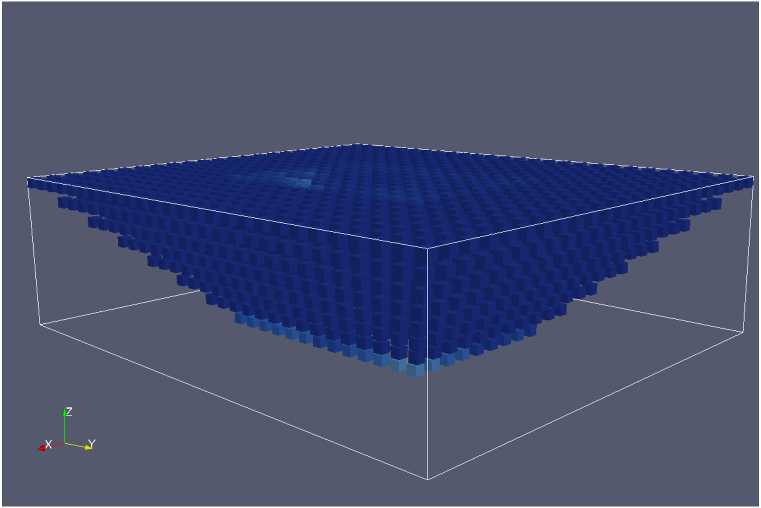
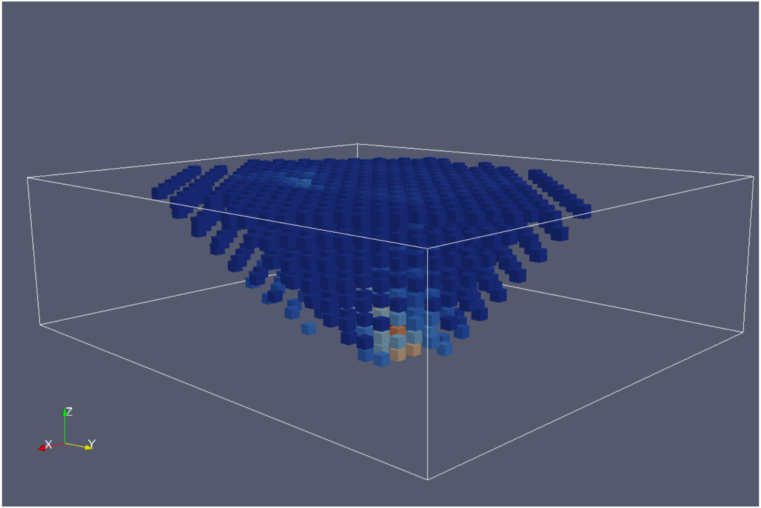
Open-Pit Mine (Chuquicamata, Chile)

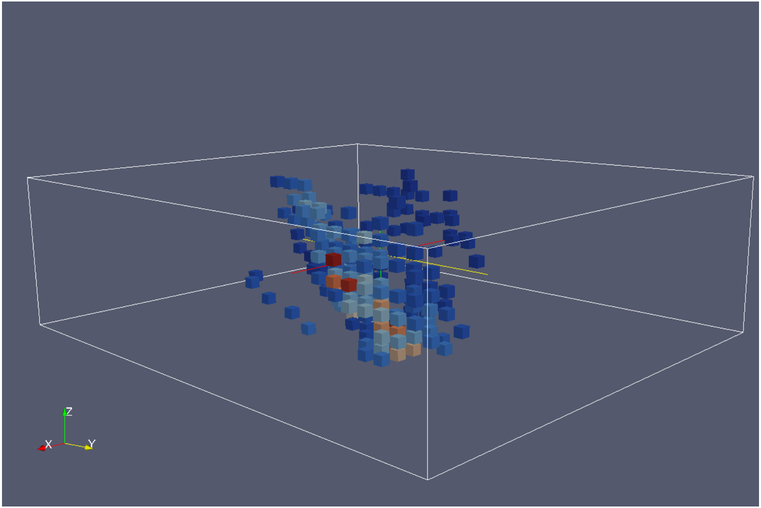
Set of blocks, colour $\sim$ amount of mineral



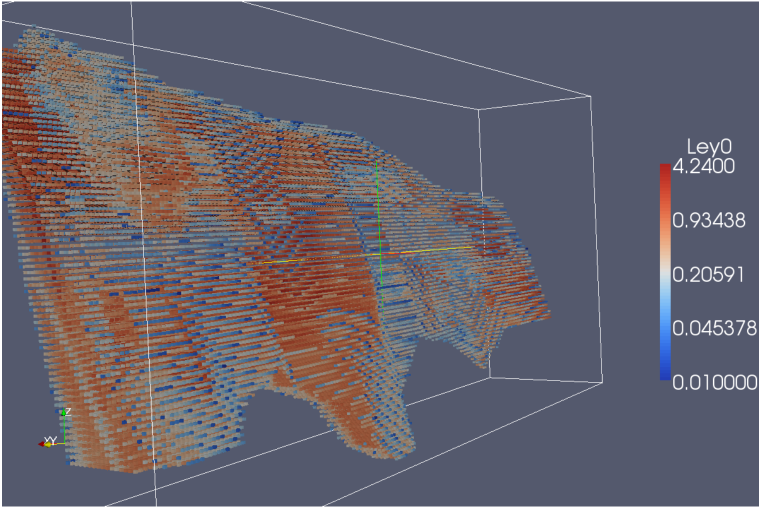
Which blocks extract?



From extracted ones, which blocks process?



Vein-type mine (medium size ∼700,000 blocks)



Characteristics

- Open-pit mine
- Planning horizon: 1 period
- Production decision is divided in:
  1. Which blocks to extract
  2. Between the extracted blocks, which to process with *process 1*, which to proces with process 2, and which not to process at all
- Processing is *exclusive*: each extracted block can be processed with *process 1* or with *process 2* but not with both

Formulation

Maximize Profit function:

$$\Pi_{\rho}(x^e, x^{p_1}, x^{p_2}) := \sum_{b\in\mathcal{B}} \Big[\; - v_b^e \, x_b^e + (w_b^{p_1}\rho_b - v_b^{p_1})\,(x_b^{p_1} - x_b^{p_2}) + (w_b^{p_2}\rho_b - v_b^{p_2})\, x_b^{p_2} \; \Big]$$

Subject to $(x^e, x^{p_1}, x^{p_2}) \in \{0,1\}^{3\cdot|\mathcal{B}|}$ is a factible extracting and processing plan:

$$\begin{array}{lll}
x_a^e \le x_b^e & \forall (a,b) \in \mathcal{P} & : \text{extr. precedences} \\
x_b^{p_2} \le x_b^{p_1} \le x_b^e & \forall b \in \mathcal{B} & : \text{exclusive pr.} \\
c^e \cdot x^e \le B^e & & : \text{extr. capacity} \\
c^{p_1} \cdot (x^{p_1} - x^{p_2}) \le B^{p_1} & & : \text{pr. 1 capacity} \\
c^{p_2} \cdot x^{p_2} \le B^{p_2} & & : \text{pr. 2 capacity}
\end{array}$$

- Precedence-constrained 0-1 knapsack problem

- We'll consider that the **ore grade of each block is an uncertain parameter:** $\widetilde{\rho}_b$
  - $\rightarrow$ This only affects profit function: $\Pi(x^e, x^{p_1}, x^{p_2}, \widetilde{\rho})$
  - $\rightarrow$ Π is affine linear in $\widetilde{\rho}$
- We have joint ore grade distribution $(\widetilde{\rho}_b)_{b\in\mathcal{B}} \rightsquigarrow \Lambda$ that can be sampled
- We have an i.i.d. sample of it:

$$\rho^1, \ldots, \rho^N \quad \text{such that} \quad \mathbb{P}\left(\widetilde{\rho} = \rho^i\right) = \frac{1}{N} \quad \forall i$$

  - $\rightarrow$ i.i.d. sampling supports covariance and weighted sampling
- **We want to compare different approaches of robustness to deal with uncertainty in the ore grades**

Value-at-Risk Model I

For a risk level $\epsilon \in [0, 1)$ (small) we'd like to solve the chance constrained model:

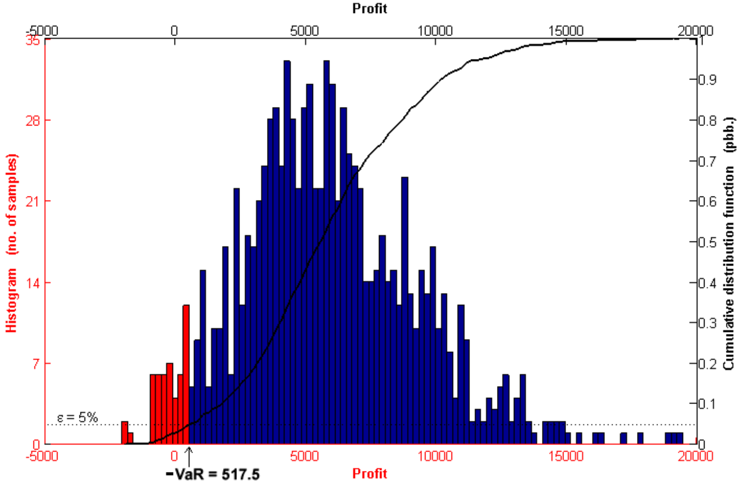
$$\begin{cases} \underset{x\,\zeta}{\text{Max}} & \zeta \\ \text{s.t.} & \mathbb{P}_{\widetilde{\rho}}\left[\, \Pi\,(x^e, x^{p_1}, x^{p_2}, \widetilde{\rho}) \geq \zeta \,\right] \geq 1 - \epsilon \\ & (x^e, x^{p_1}, x^{p_2}) \in X, \qquad \zeta \in \mathbb{R} \end{cases}$$

- Equivalent to minimize **Value-at-Risk** (VaR) of the Profit Π

$$\text{VaR}_\epsilon(Y) := -\text{Max}\,\{t \in \mathbb{R} \;:\; \mathbb{P}(Y \geq t) \geq 1 - \epsilon\} = -(\epsilon\text{-percentile for } Y)$$

- (Approximation) Formulation: MILP with “Big-M”
- Sampled approximation is consistent under *mild* assumptions

Value-at-Risk Model II



Processing decision as a Recourse variable

We'd also like to consider the next decision scheme:

1. We first decide which blocks to **extract**
2. Once extracted, **we can see the real ore grade** of each extracted block
3. We then decide **which extracted blocks to process** and how

So, basically, the **processing decision is a *Recourse variable***

- Formulation: put one processing decision per scenario
- Recourse variant of VaR model is also consistent

Conditional Value-at-Risk Model I

Conditional Value-at-Risk (CVaR) for risk level $\epsilon \in (0, 1]$:

- If Y (profits) has atomless distribution,

$$\mathrm{CVaR}_\epsilon(Y) := -\mathbb{E}[\ Y \mid Y \leq \underbrace{-\mathrm{VaR}_\epsilon(Y)}_{\epsilon\text{-percentile for } Y}\ ]$$

- CVaR model:

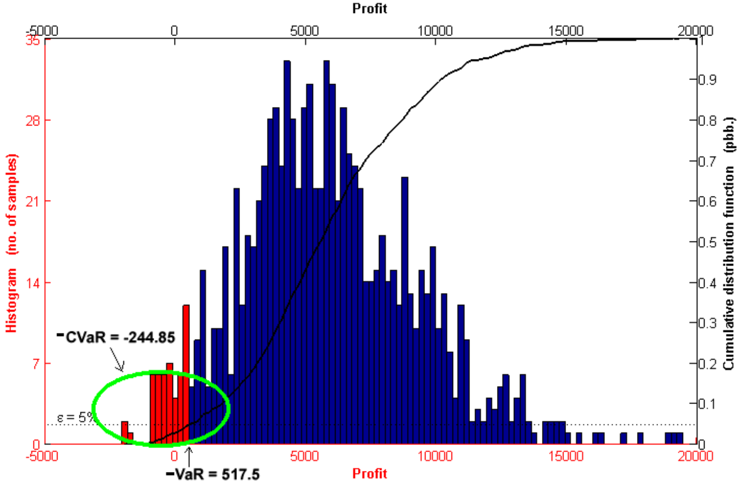
$$\begin{cases} \min & \mathrm{CVaR}_\epsilon\,[\,\Pi(x^e, x^{p_1}, x^{p_2}, \widetilde{\rho})\,] \\ \text{s. t.} & (x^e, x^{p_1}, x^{p_2}) \in X \end{cases}$$

Has MILP formulation. Model with recourse too.

! Case $\epsilon = 1$ equivalent to $\underset{(x^e, x^{p_1}, x^{p_2}) \in X}{\mathrm{Max}} \mathbb{E}\,[\Pi(x^e, x^{p_1}, x^{p_2}, \widetilde{\rho})]$

- Sampled approximation, with and w/o recourse, **is consistent.**

Conditional Value-at-Risk Model II



Modulated Convex-Hull Model I

We propose the next *uncertainty-set based* robust model:

$$\begin{cases} \underset{x}{\text{Max}} & \min_{\rho \in \mathcal{U}_\epsilon} \Pi(x^e, x^{p_1}, x^{p_2}, \rho) \\ \text{s. t.} & (x^e, x^{p_1}, x^{p_2}) \in X \end{cases}$$

where $\epsilon \in [0, 1]$ is a desired risk level and

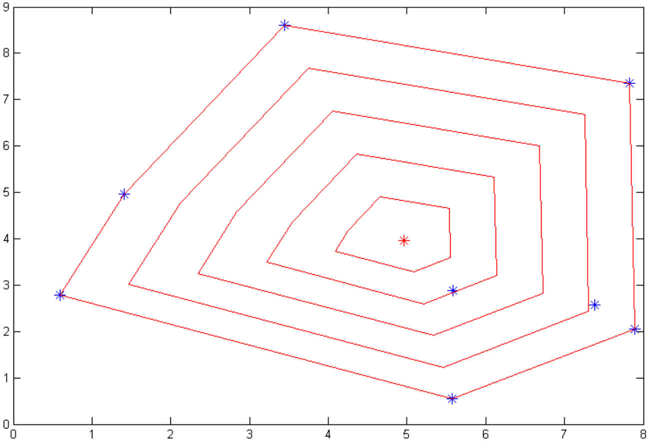
$$\mathcal{U}_\epsilon = \bar{\rho} + (1 - \epsilon) \left(\text{conv} \left\{ \rho^1 \dots \rho^N \right\} - \bar{\rho} \right)$$

- Has MILP formulation. Model with Recourse too.

! Case $\epsilon = 1$ equivalent to $\underset{(x^e, x^{p_1}, x^{p_2}) \in X}{\text{Max}} \mathbb{E} [\Pi(x^e, x^{p_1}, x^{p_2}, \widetilde{\rho})]$

Modulated Convex-Hull Model II

Example: $\mathcal{U}_\epsilon$ for $N = 8$ samples of $\widetilde{\rho} \in \mathbb{R}^2$



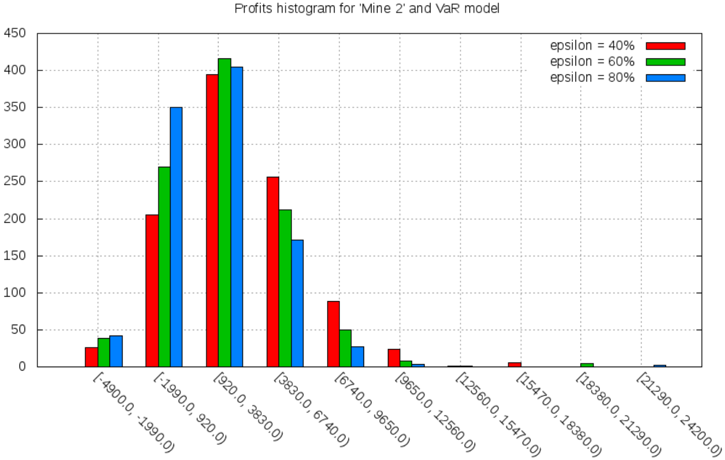
Modulated Convex-Hull Model III

Proposition: **Mod. Convex Hull as risk averse optimization**

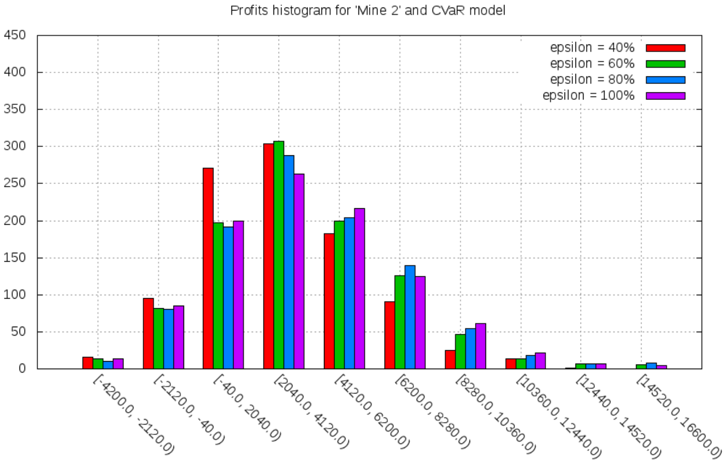
1. $$\max_{(x^e, x^{p_1}, x^{p_2}) \in X} \min_{\rho \in \mathcal{U}_\epsilon} \Pi(x^e, x^{p_1}, x^{p_2}, \rho)$$
$$\iff \min_{(x^e, x^{p_1}, x^{p_2}) \in X} \mu_{\epsilon,N}\left(\Pi(x^e, x^{p_1}, x^{p_2}, \widetilde{\rho})\right)$$
   where $\mu_{\epsilon,N}$ is the risk measure
$$\mu_{\epsilon,N}(Z) = (1-\epsilon)\underbrace{\mathbb{E}(-Z)}_{\mathrm{CVaR}_1(Z)} + \epsilon \underbrace{\max_{i=1\ldots N}\left\{-Z(\omega^i)\right\}}_{\mathrm{CVaR}_{\frac{1}{N}}(Z)}$$
   for Z r.v. in the equiprobable space $\Omega = \left\{\omega^1 \ldots \omega^N\right\}$
2. $\mu_{\epsilon,N}$ is a *distortion risk measure*: it's a *coherent risk measure*, *comonotonic* and *law invariant*.

- We consider two vein-type mines, only one process available
- For each mine we have 1000 posible scenarios for the grades
- We take a random sample of $N = 10,\ 20, ...$ out of the 1000 posible grade scenarios,
- and for risk levels $\epsilon = 10\%,\ 20\%, \ldots, 100\%$ we solve the robust models. We obtain a production plan x^{e*}, x^{p_1*} per model.
- For each model, risk level ϵ and sample size N we present the distribution of the profits:
  empiric distribution for $\Pi(x^{e*}, x^{p_1*}, \rho^k) \quad k = 1 \ldots 1000$

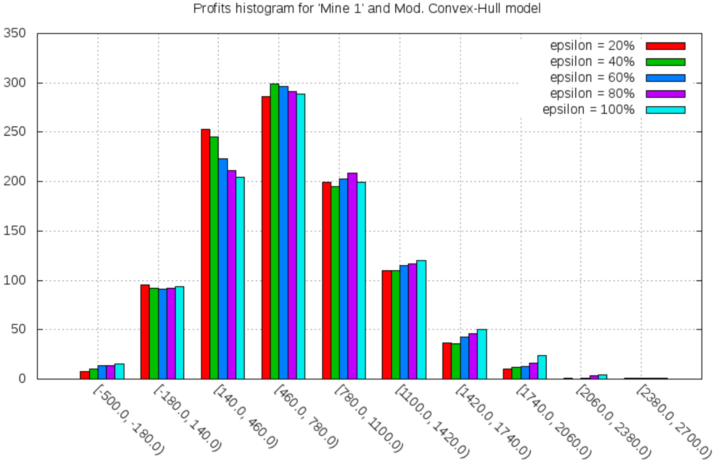
VaR model performance



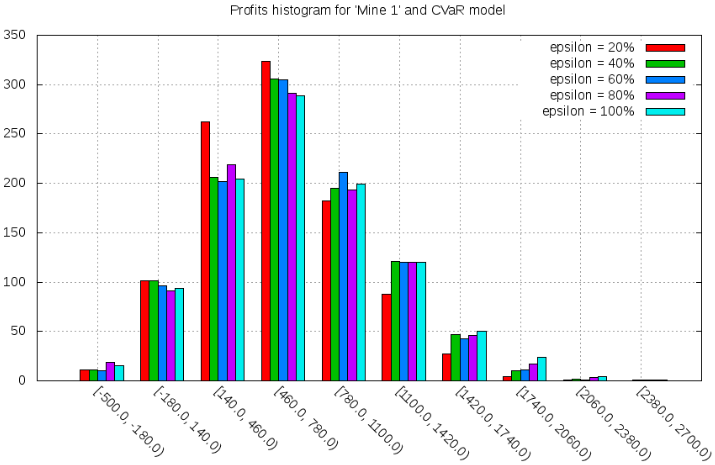
CVaR model performance



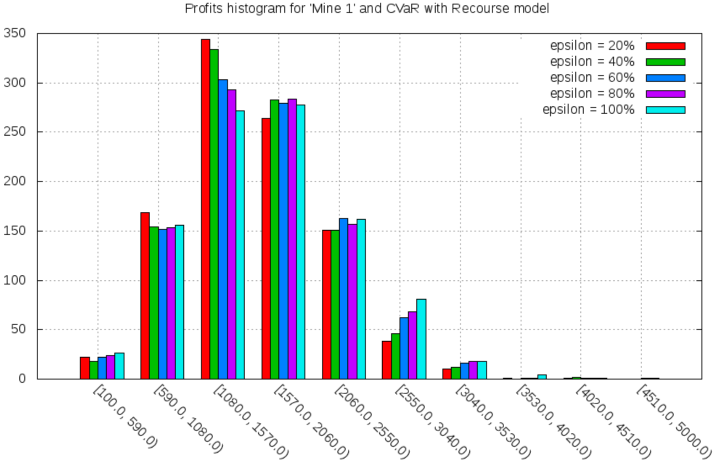
Modulated Convex-Hull model



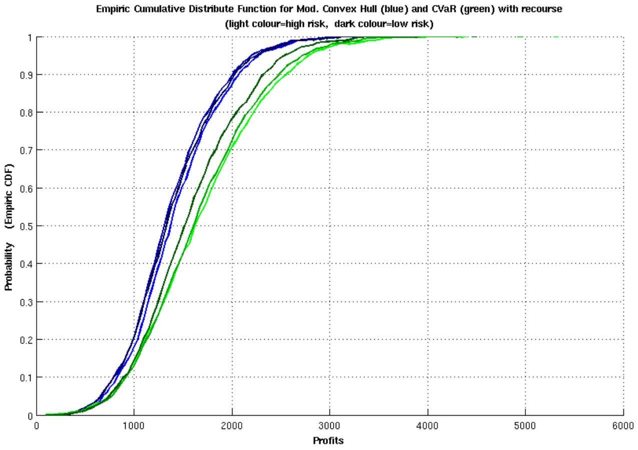
CVaR model results



CVaR with Recourse model results



Mod. Convex Hull & CVaR models with Recourse



Conclusions & Discussion

- CVaR performs considerably better than VaR in this problem
- Mod. Convex Hull model performs similar to CVaR, although they're motivated by different schemes
- Value of information: taking processing variables as recourse variables has considerable benefits
- In recourse models, CVaR shows a "more coherent" behaviour than Mod. Convex Hull
- Validation of Mod. Convex Hull model: for $\widetilde{\rho}, \rho^1 \ldots \rho^N$ i.i.d., what is $\mathbb{P}\left[\widetilde{\rho} \in \text{conv}\left\{\rho^1 \ldots \rho^N\right\}\right]$?

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Formulation of deterministic problem

$$\left\{ \begin{array}{rl} \text{Max} & \Pi(x^e, x^{p_1}, x^{p_2}, \rho) \\ \text{s.t.} & (x^e, x^{p_1}, x^{p_2}) \in X \end{array} \right.$$

- $\sim$ $3|\mathcal{B}|$ binary variables
- $\sim$ $|\mathcal{P}| + 2|\mathcal{B}| + 3$ restrictions

S.A.A. formulation of Value-at-Risk model

$$
\begin{cases}
\underset{x\,z\,\zeta}{\text{Max}} & \zeta \\
\text{s.t.} & \Pi(x^e, x^{p_1}, x^{p_2}, \rho^k) + z_k M^k \geq \zeta \quad \forall k = 1 \dots N \\
& \sum_{k=1}^{N} z_k \leq \lfloor N\epsilon \rfloor \\
& z_k \in \{0,1\}^N \quad \forall k = 1 \dots N \\
& \zeta \in \mathbb{R} \\
& (x^e, x^{p_1}, x^{p_2}) \in X
\end{cases}
$$

for sufficiently big $M^k \in \mathbb{R}$ parameter

- N binary variables & 1 continuum variable
- $(N+1)$ restrictions

S.A.A. formulation of Conditional Value-at-Risk model

$$
\left\{
\begin{array}{lll}
\min\limits_{x\,\zeta\,\eta} & \zeta + \frac{1}{N\epsilon}\sum\limits_{k=1}^{N}\eta_k & \\
\text{s.t.} & \Pi\left(x^e, x^{p_1}, x^{p_2}, \rho^k\right) + \zeta + \eta_k \geq 0 & \forall k = 1\ldots N \\
& \eta_k \geq 0 & \forall k = 1\ldots N \\
& \zeta \in \mathbb{R} & \\
& (x^e, x^{p_1}, x^{p_2}) \in X &
\end{array}
\right.
$$

+ $(N+1)$ continuum variables
+ $2N$ restrictions

S.A.A. formulation of Mod. Convex-Hull model

$$\begin{cases} \underset{x\,\zeta}{\text{Max}} & \zeta \\ \text{s.t.} & \Pi(x^e, x^{p_1}, x^{p_2}, \bar{\rho} + (1-\epsilon)(\rho^k - \bar{\rho})) \geq \zeta \quad \forall k = 1 \dots N \\ & \zeta \in \mathbb{R} \\ & (x^e, x^{p_1}, x^{p_2}) \in X \end{cases}$$

+ 1 continuum variable
+ N restrictions

Consistency of S.A.A. for VaR model

Proposition 5.30, [SPBook]

If there exists a solution $(x^{e*}, x^{p_1*}, x^{p_2*}, \zeta^*)$ for the real problem

$$\theta^* := \begin{cases} \underset{x\,\zeta}{\text{Max}} & \zeta \\ \text{s.t.} & \mathbb{P}_{\widetilde{\rho}}\,[\,\Pi\,(x^e, x^{p_1}, x^{p_2}, \widetilde{\rho}) \geq \zeta\,] \geq 1 - \epsilon \\ & (x^e, x^{p_1}, x^{p_2}) \in X, \qquad \zeta \in \mathbb{R} \end{cases}$$

such that $\forall \epsilon > 0$ there exists a factible solution $(x^e, x^{p_1}, x^{p_2}, \zeta)$ to the true problem that satisfies

- $||(x^e, x^{p_1}, x^{p_2}, \zeta) - (x^{e*}, x^{p_1*}, x^{p_2*}, \zeta^*)|| \leq \epsilon$
- $\mathbb{P}_{\widetilde{\rho}}\,[\,\Pi\,(x^e, x^{p_1}, x^{p_2}, \widetilde{\rho}) \geq \zeta\,] > 1 - \epsilon$

then the approximated estimator is consistent and the distance between the set of optimal solutions of the approximated problem and the optimal solutions of the true problem tends to 0.