Discretized Brownian motion about random times

Limit distributions and random walk approximations

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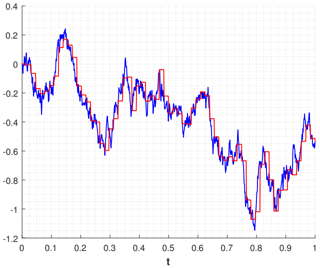
Motivation: simulation of Brownian motion

- Brownian motion — fundamental stochastic process, in theory and practice.

$\Rightarrow$ **Relevant!**

- Simulation of Brownian motion
  - continuous time process, violent fluctuations, “self-similar” in time-space
  - $\Rightarrow$ **Challenging!**
  - Some methods: exact simulation of some specially structured events; approximation on time mesh; approximation on space mesh; wavelet approximations; so-called ϵ-strong approximations; to name a few ...

Setting and goal



Setting: Euler discretization

- Brownian motion B on $\mathbb{R}$
- Time mesh $\{0,\ 1/n,\ 2/n,\ \ldots\}$
- B^n := piecewise constant approx. of B on time mesh

$*$ $B(t+h) - B(t) \sim N(\mu h, \sigma^2 h)$. **Easy!**

- PROS ☺: simple / efficient / sensible in some applications
- CONS ☹: little control on accuracy

Goal:

T := time of min of B on $[0, 1]$
T^n := same but for approximation B^n

Analyze time error $T^n - T$ and position error $B^n(T^n) - B(T)$ as $n \to \infty$.

Also do same error analysis when T is

- time of min of B on $[0, \infty)$
- first time B hits or goes above "barrier" function b

Main results I

Theorem (Euler discretization weak limits)

B Brownian motion on $\mathbb{R}$, with constant drift μ and unit variance,

$$T := \text{time of minimum of } B \text{ on time interval } [0,1].$$

B^n Euler discretization of B on mesh $\{0, \frac{1}{n}, \frac{2}{n}, \ldots\}$,

$$T^n := \text{time of minimum of } B^n \text{ on time interval } [0,1].$$

It holds that

$$\begin{array}{c} \sqrt{n}\,(B^n(T^n) - B(T)) \\ n\,(T^n - T) \end{array} \quad \underset{n\to\infty}{\Longrightarrow} \quad \begin{array}{rr} \min_{k\in\mathbb{Z}} R(U+k) & \text{[AGP 1995]} \\ U + \arg\min_{k\in\mathbb{Z}} R(U+k), & \text{[us]} \end{array}$$

with R two-sided Bessel(3) process, U independent uniform[0,1].

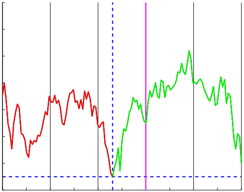
Main results II

Corollary (Gaussian walk limits)

Gaussian walk S with drift $\mu = 0$ and unit variance.
Brownian motion B such that $B(k) = S_k$ for all $k \in \mathbb{N}$.

$$\begin{matrix} \min_{0\le k\le n} S_k - \min_{0\le t\le n} B(t) \\ \arg\min_{0\le k\le n} S_k - \arg\min_{0\le t\le n} B(t) \end{matrix} \underset{n\to\infty}{\Longrightarrow} \begin{matrix} \min_{k\in\mathbb{Z}} R(U+k) \\ U + \arg\min_{k\in\mathbb{Z}} R(U+k) \end{matrix}$$

with R two-sided Bessel(3) process, and U independent uniform[0,1].



- Intuition: Williams-type decomposition of Brownian motion at minimum
- Renewal theory flavor. **GIF!**
- Gaussian walk $\approx$ Brownian motion. *Diffusion approximation.*

Main results III: a collection of results

1. The following converge in distribution to the pair

$$\left(U + \arg\min_{k\in\mathbb{Z}} R(U+k)\ , \quad \min_{k\in\mathbb{Z}} R(U+k) \right)$$

   where U uniform[0,1], R two-sided Bessel(3) process:
   - $(n$ (time error), $\sqrt{n}$ (position error)$)$ for local min of Brownian motion B v/s same on mesh $\{0,\ 1/n,\ 2/n,\ \ldots\}$, as $n \to \infty$.
   - (time error , position error) for local min of Gaussian walk v/s same for Brownian motion, on time horizon $[0, n]$, as $n \to \infty$.
   - $(n$ (time error), $\sqrt{n}$ (position error)$)$ for global min of Brownian motion B with positive drift v/s same on mesh $\{0,\ 1/n,\ 2/n,\ \ldots\}$, as $n \to \infty$.
   - (time error , position error) for global min of Gaussian walk v/s same for Brownian motion, both with drift $\mu > 0$, as $\mu \searrow 0$.
2. The following converge in distribution to the pair

$$(\ U + k^*\ , \quad W(U+k^*)\ )$$

   where U uniform[0,1], W standard BM, $k^* := \min\{k \in \mathbb{Z}_+ : W(U+k) > 0\}$:
   - $(n$ (time error), $\sqrt{n}$ (position error)$)$ for hitting time of B to a non-decreasing barrier v/s same on mesh $\{0,\ 1/n,\ 2/n,\ \ldots\}$, as $n \to \infty$.
   - (time error , position error) for barrier-hit of Gaussian walk v/s same for Brownian motion, to hit barrier $b > 0$, as $b \nearrow \infty$.

A constant in the literature...

For $\zeta =$ Riemann zeta function,

$$-\frac{\zeta(1/2)}{\sqrt{2\pi}} \underset{\text{AGP '95}}{=} \mathbb{E}\left[\min_{k\in\mathbb{Z}} R(U+k)\right] \underset{us}{=} \mathbb{E}\left[W(U+k^*)\right].$$

Constant $-\zeta(1/2)/\sqrt{2\pi}$ appears dispersed in literature:

- Sequential analysis: Chernoff (1965), Siegmund (1985)
- Diffusion approximations: Siegmund (1979), ...
- Euler discretization: Asmussen, Glynn & Pitman (1995), Calvin (1995)
- Option pricing: Broadie, Glasserman & Kou (1997), (1999), ...
- Gaussian walks: Chang & Peres (1997), Janssen & van Leeuwaarden (2007)^2, (2013), ...

Connection was unknown... **Our results connect all these papers**

Summary & main contributions

1. We derive limit distributions for normalized discretization errors. Analytical, closed form.
2. We put limits into context of Brownian motion approximation of Gaussian walk. These are new results in the theory of diffusion approximation.
3. We provide a unified framework connecting several papers in the literature where the constant $-\zeta(1/2)/\sqrt{2\pi}$ appears.

Thanks!